Homogeneous Geometry Calculation of Arbitrary Tooth Shapes: Mathematical Approach and Practical Applications

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This paper provides a mathematical framework and its implementation for calculating the tooth geometry of arbitrary gear types, based on the basic law of gear kinematics. The rack or gear geometry can be generated in two different ways: by calculating the conjugate geometry and the line of contact of a gear to the given geometric shape of a known geometry (e.g., a cutting hob), or by prescribing the surface of action of two gears in contact and calculating the correspondent flank shapes. Besides so-called standard gears like involute spur and helical gears, bevel or worm gears, it is possible to analyze the tooth geometry of non-standard gears (e.g., non-involute spur, conical, or spiroid gears). Depending on the type of gear, a distinction is made between tool-dependent and tool-independent geometry calculation.

As an extensive machine element to transfer and convert rotational movement, gears meet high requirements for construction and assembly. Due to existing modern production techniques, more sophisticated gear types can be produced with high precision and maintainable financial effort. The benefits of traditional gear profiles, such as an involute, are thus no longer of major importance. In particular, for gear types such as bevel, worm, and hypoid gears, but also for non-standard gear types (e.g., beveloid gears, crown gears, or spiroid gearings), modern gear production systems ensure high quality and reliability to the operator. Depending on the context of application, different gear types have advantages and disadvantages concerning load carrying capacity, effectiveness, or noise excitation. Supported by various calculation software tools for the particular gear type, it is possible to create the optimal gear design, depending on the respective application. A homogeneous calculation software for ubiquitous gear geometries—irrespective of the gear type, and especially for analyzing non-standard gears—would be preferable. This paper provides a mathematical framework and its implementation for calculating the tooth geometry of arbitrary gear types, based on the basic law of gear kinematics. The rack or gear geometry can be generated in two different ways: by calculating the conjugate geometry and the line of contact of a gear to the given geometric shape of a known geometry (e.g., a cutting hob), or by prescribing the surface of action of two gears in contact and calculating the correspondent flank shapes. Besides so-called standard gears like involute spur and helical gears, bevel or worm gears, it is possible to analyze the tooth geometry of non-standard gears (e.g., non-involute spur, conical, or

NOMENCLATURE

Symbol	Description	Unit
a	center distance	mm
b	face width	mm
C_β	lead crowning	µm
d	reference diameter	mm
d_b	base diameter	mm
$d_{(F)a}$	(utilized) tip diameter	mm
$d_{(F)f}$	(utilized) root diameter	mm
m_n	normal module	mm
m_t	transverse module	mm
m_x	axial module	mm
u	gear ratio	-
x	addendum modification coefficient (profile shift)	-
z	number of teeth	-
α_n	normal pressure angle	°
β	helix angle	°
γ	worm pitch angel	°
Δ	gear rotation angle for uniform motion transfer	°
Δ_x	local gear rotation angle for non-uniform motion transfer	°
Δ	gear angular velocity	°/s
η	worm swivel angle	°
θ	addendum modification angle (cone angle)	°
Θ	auxiliary angle for calculating parallel sections worm profile	°
v	translational rack velocity	mm/s
φ	arbitrary rotation angle	°
ψ	tool displacement angle (beveloid manufacturing process)	°
Σ	axis crossing angle	°
a_i	axis point vector	
b_i	axis directional vector	
c_i	arbitrary vector	
n_i	outer unit normal vector	
x_i	vector of point of contact	
x	vector of circumferential velocity	
y_i	vector / surface / point cloud of a tooth shape	
A, B, C	scalars to simplify formulas of conjugate gearing	
$D_b(\varphi)c$	rotation function of vector c around axis b by angle of j	
τ	face width parameter of beveloid/cylindrical gears	
σ	profile parameter of cylindrical gears	
1	index of gear 1 respectively index of rack	
2	index of gear 2 respectively index of work piece	
0	index of tool (e.g. grinding worm or hob)	

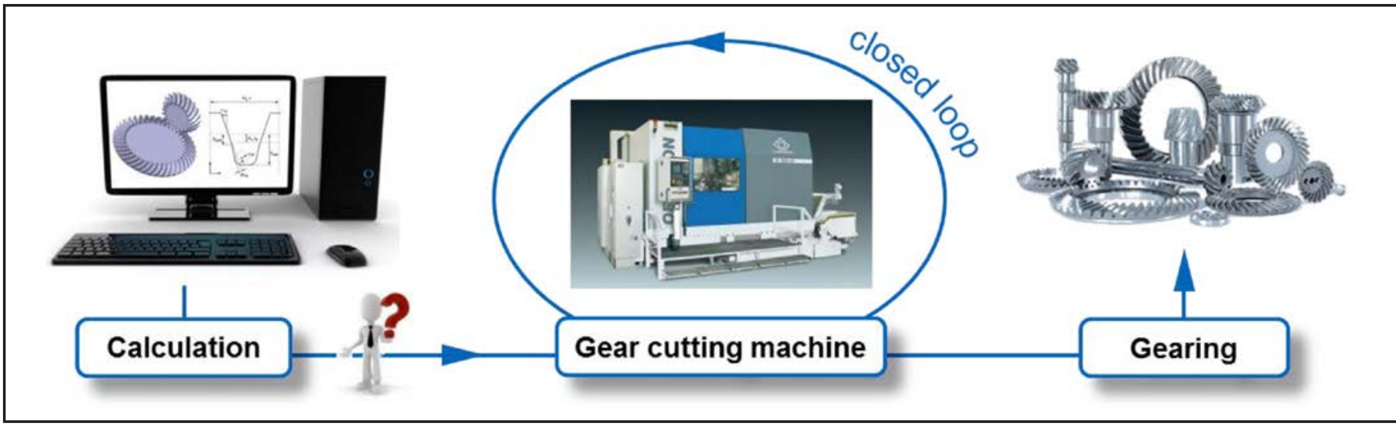


Figure 1 Process chain for an optimized integrated calculation and manufacture of arbitrary gears (Graphic courtesy Klingelnberg GmbH).

spiroid gears). Depending on the type of gear, a distinction is made between tool-dependent and tool-independent geometry calculation. The described mathematical algorithms are summarized in implemented software modules for the particular gear types. Two practice-oriented examples are presented to illustrate the calculation model: beveloid gears for use in vehicle or marine gear boxes as well as rack-and-pinion meshing with variable ratio, as it is used for steering systems on automobiles. Since the geometry is exported as a point cloud, a further analysis of the generated gear types is possible, e.g., by computer-aided design or finite- element software tools as well as manufacturing on 5-axis CNC or forging machines. Thus, a detailed analysis — especially of non-standard gears — is feasible that currently cannot be calculated and evaluated with common industrial gear calculating software. The project is funded by the Forschungsvereinigung Antriebstechnik e.V. (FVA).

Motivation

Gears are complex machine elements designed to transfer and convert rotational or translational movement. Engineers face high requirements in construction and assembly to design gears that reach proper function, in particular with regard to cost effectiveness and sufficient load capacity.

Due to existing modern production techniques, more sophisticated gear types can be produced with high precision and maintainable financial effort. Thus, the benefits of traditional gear profiles — such as an involute — are no longer of major importance. Especially for non-standard gear types — beveloid gears, face gears or spiroid gearings, for example — modern gear production systems ensure high quality and reliability to the operator. The tooth shape geometry of these gears has a significant impact on the competing goals of efficiency, load capacity, and noise excitation. To achieve an optimized gear profile, depending on the respective application, it is inevitable that production needs to be supported by high-performance calculation software.

Therefore, it is necessary in the first design step to develop a uniform mathematical framework for an integrated calculation of arbitrary gear types in any axis position. The algorithms should be able to provide any kind of three-dimensional tooth geometry fast and analytically. The main focus is the computation of practice-oriented gears with regard to the concrete manufacturing kinematics of a gear cutting machine. Besides, the potentials of new gear shapes can be tapped without the restriction of a common manufacturing machine. By numerical generation of the gear shape via point cloud, further analyses are possible, e.g., by computer-aided design or finite-element software tools, in addition to manufacturing on 5-axis CNC or forging machines.

To sum up: a fast and analytical uniform calculation method and its implementation would help to support the integrated gear design and generation process of any kind of gearings (Fig. 1).

Mathematical Fundamentals of Tooth Geometry Calculation

Starting from the fundamental demand for a continuous uniform motion transfer with constant gear ratio u, all mathematical principles can be described by the basic law of gear kinematics for conjugate shapes. In particular, this means that the angular velocity $\Delta_{1,2}$ of both gears is directly proportional to the number of teeth z_2/z_1:

$$\frac{z_2}{z_1}=\frac{\dot{\Delta}_1}{\dot{\Delta}_2}=u \tag{1}$$

To keep the flanks staying in contact, the vector of relative motion of both flanks has to be perpendicular to the outer unit normal vector. Otherwise, the touching flanks would pass through each other or lose contact. That means, in particular, that the normal velocity of both flanks has to be commensurate in any point of contact. Gear 1 is mounted on axis $a_1+\lambda\cdot b_1$ and gear 2 on axis $a_2+\mu\cdot b_2$. The basic law of gear kinematics becomes (Ref. 1):

$$0=\langle n_i, x_1-x_2\rangle=\left\langle n_i, \left(b_1\times(x_1-a_1)-\frac{z_1}{z_2}\cdot b_2\times(x_1-a_2)\right)\right\rangle \tag{2}$$

with the point vector a_i and directional vector b_i. In Equation 2, x_i is the contact point of both flanks. Therefore, Figure 2 shows schematically the relations of the basic law of gear kinematics for gear-gear meshing.

As the basic law of gearing is defined, we are now able to calculate the conjugate flank to a given shape. Starting from a given surface parametrization y_1 (or computed point cloud, such as a 3-D measurement of a gear) with the corresponding outer unit normal vectors ny_1, we first rotate the given shape

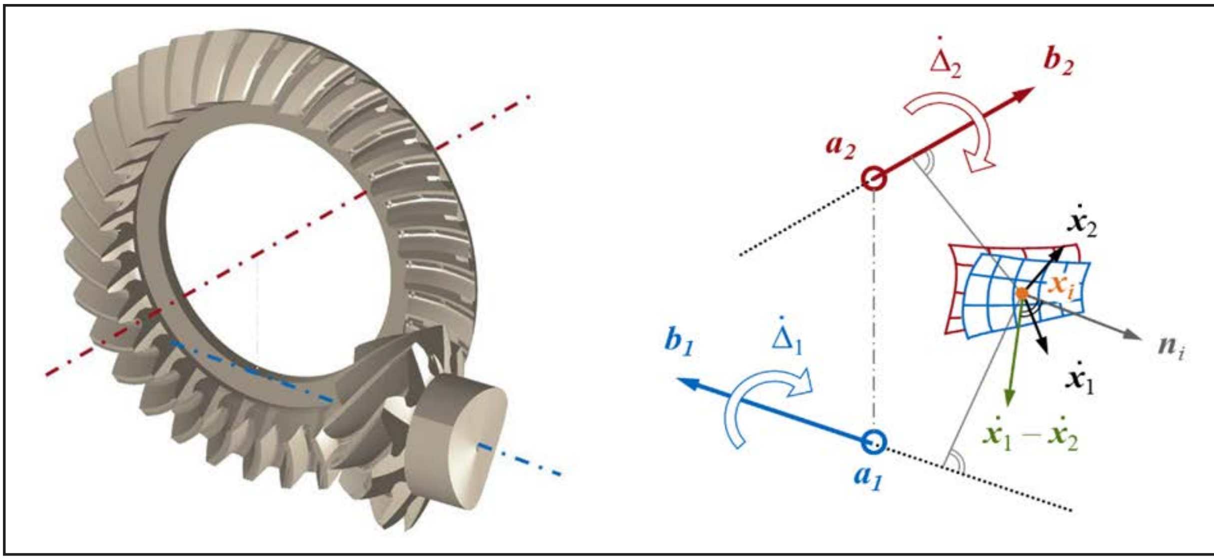


Figure 2 On the basic law of gear kinematics for gear-gear meshing.

by the (so far unknown) angle Δ to solve Equation 2. The rotated point thus becomes the current point of contact

$$x_1 = x_2 = a_1 + D_{b_i}(\Delta)(y_1 - a_1) \tag{3}$$

$D_{b_i}(\varphi_i)c_i$ denotes a rotation of vector c_i by the angle φ_i around axis b_i (Ref. 1). The desired shape y_2 can be computed by rotating the point of contact x around the axis of gear 2 by the angle of $-\Delta z_1/z_2$:

$$y_1 = a_2 + D_{b_2}\left(-\Delta \cdot \frac{z_1}{z_2}\right)(x_1 - a_2) \tag{4}$$

The angle Δ to solve Equation 2 is presented by (Ref. 1):

$$0 = A + B\cos\Delta + C\sin\Delta \tag{5}$$

with the scalars (Ref. 1):

$$A = \langle n_{y1}, b_1 \times (y_1 - a_1)\rangle - \frac{z_1}{z_2}\langle n_{y1}, b_1\rangle \cdot \langle b_1, b_2 \times (a_1 - a_2)\rangle + \frac{z_1}{z_2}\langle b_2, b_1\rangle \cdot \langle b_1, n_{y1}, \times (y_1 - a_1)\rangle \tag{6}$$

$$B = \frac{z_1}{z_2} \cdot \left(n_{y1} - \langle n_{y1}, b_1\rangle \cdot b_1, b_2 \times (a_1 - a_2)\rangle + \langle b_2 - \langle b_2, b_1\rangle \cdot b_1, n_{y1} \times (y_1 - a_1)\rangle \right)$$

$$C = \frac{z_1}{z_2} \cdot \left(\langle n_{y1} \times b_1, b_2 \times (a_1 - a_2)\rangle + \langle b_2 \times b_1, n_{y1} \times (y_1 - a_1)\rangle \right)$$

We solve Equation 5 by usage of the *arctan*2-function:

$$\Delta = \operatorname{arctan2}(y, x) \text{ with co-domain } \pi < \operatorname{arctan2}(y, x) \leq \pi \tag{7}$$

and the arguments (y,x) (Ref. 1)

$$(y, x) = \left(-A \cdot C \mp B \cdot \sqrt{B^2 + C^2 - A^2}, -A \cdot B \pm C \cdot \sqrt{B^2 + C^2 - A^2}\right. \tag{8}$$

As the algorithms have to be implemented in a selected computer language, the purpose of using two arguments in the *arctan*2-function instead of one is to return the appropriate quadrant of the computed angle, which is not possible for the single-argument *arctan*-function (Ref. 2). It can be easily seen that Equation 5 has two solutions in which the solution with min $\|\dot{x}_1 - \dot{x}_2\|$ is the one preferred for practice-oriented gear shapes (Ref. 1).

As mentioned in the opening, another generation method calculates a conjugate pair of flanks from a given surface of action with definition of an initial line of contact (Ref. 1). Since this algorithm is difficult to apply to the simulation of a gear manufacturing process (the surface of action between tool and workpiece is generally not known before the generation process, but results as an outcome of it), it is not used in the applications presented in this paper. For further information, see (Refs. 1 and 3).

Besides the meshing of two (axially symmetric) gears, gear rack engagement is relevant — especially for the generation process of cylindrical or conical gears — as well as worm gears or rack-and-pinion meshing in general. Therefore, adhering to enhancement of the basic law of gear kinematics is necessary.

The rack is given by a shape with translational velocity v_1 moving in the direction of b_1 (Fig. 3). The gear is rotating with angle Δ_2 around its axis b_2. With the point of contact x_i and unit normal vector n_i, the basic law for gear rack meshing becomes (Ref. 3):

$$\langle n_i, v_1 \cdot b_1 - b_2 \times (x_i - a_2)\rangle = 0 \tag{9}$$

For applications such as generation of a gear by a tool shape (e.g., with a basic rack profile [Ref. 4]) we can calculate the gear shape y_2 by a given rack

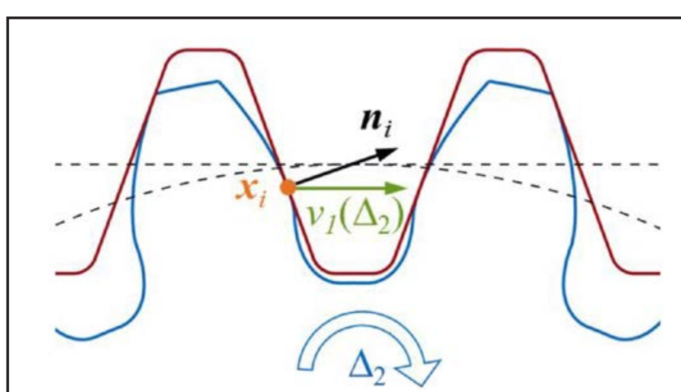


Figure 3 On the basic law of gear kinematics for gear-rack-meshing.

contour y_1 (Ref. 3):

$$y_2 = a_2 + D_{b_2}(\Delta)(y_1 + \Delta \cdot \upsilon_1 \cdot b_1 - a_2) \tag{10}$$

The angle Δ_2 to solve the basic law of gearing (Eq. 9) results analytically by (Ref. 3):

$$\Delta_2 = \frac{\langle n_i, b_2 \times b_1 \cdot \upsilon_1 \rangle}{\langle n_i, \upsilon_1 \cdot b_1 - b_2 \times (y_i - a_2) \rangle} \tag{11}$$

Furthermore, we can go the other way around and compute the tool geometry y_1 to manufacture a given gear shape y_2 (e.g., with modified, non-involute flank or root fillet contour) for optimized bending stress. To determine the conjugate rack geometry, the solution of the adapted basic law results in (Ref. 3):

$$y_1 = a_2 + D_{b_2}(\Delta_2)(y_2 - a_2) - \upsilon_1 \cdot \Delta_2 \cdot b_1 \tag{12}$$

Moreover, it is possible to extend Equation 12 for non-uniform motion transfer. With $\upsilon_1 = f(\Delta_2)$ the thus far constant ratio between rack and gear becomes a function of the gear rotation angle Δ_2 — and therefore variable. The variable, three-dimensional tooth shape of the rack can now be calculated as

$$y_1 = a_2 + D_{b_2}(\Delta_x)(y_2 - a_2) - b_1 \cdot \int_{\Delta_0}^{\Delta_x} \upsilon(\Delta) d\Delta \tag{13}$$

in which Δ_x is the angle searched for in solving the basic law (Eq. 9) subject to the rack gain $\upsilon_1(\Delta_2)$. The angle Δ_x can be calculated for each point of contact by solving:

$$0 = A(\Delta_x) + B \cdot \cos \Delta_x + C \cdot \sin \Delta_x \tag{14}$$

with the scalars:

$$A(\Delta_x) = \langle n_2, b_2 \rangle \cdot \langle b_1, b_2 \rangle - \frac{\langle n_2, b_2 \times (y_2 - a_2) \rangle}{\upsilon_1(\Delta_2)} \tag{15}$$

$$B = \langle n_2, b_1 \rangle - \langle n_2, b_2 \rangle \cdot \langle b_1, b_2 \rangle \text{ and } C = \langle b_2 \times n_2, b_1 \rangle$$

Due to the fact that Equation 14 has no analytical solution, Δ_x must be calculated by appropriate numerical methods for each "local" gear ratio $\upsilon_1(\Delta_2)$.

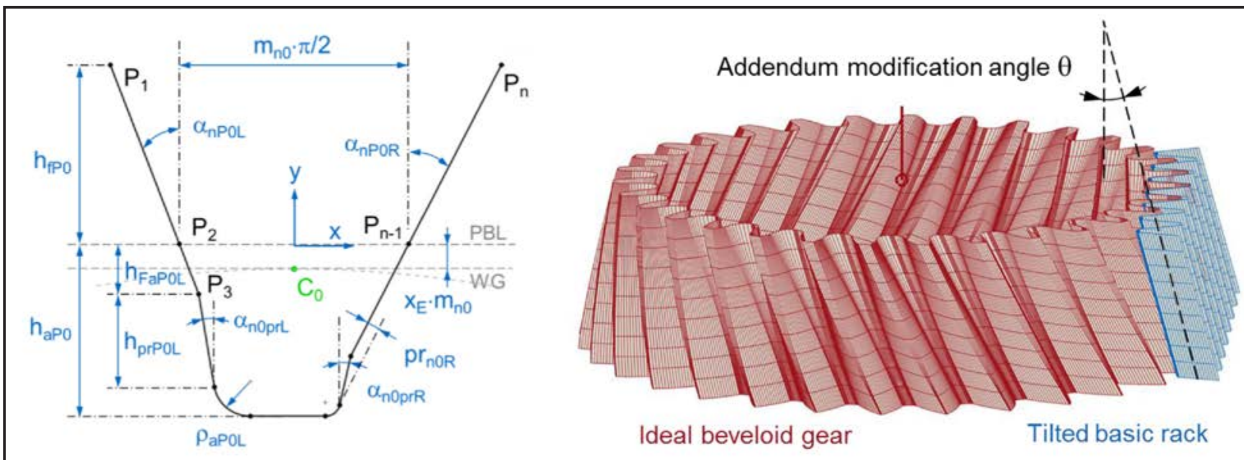
Practical Application: Generation Process of Beveloid Gears

Beveloid gears are spur or helical gears with involute profile and variable addendum modification, realized by a cone angle θ, thus getting tapered root and outside diameters. Beveloid gears can be used for example in automobile and marine applications with parallel, intersected or crossed axes — and in a variety of relative positions. As they can be manufactured on common involute, cylindrical gear cutting machines, beveloid gears assure a cost-effective alternative to bevel or hypoid gearsets — especially for small amounts of axis-crossing angle, say less than 15° (Ref. 5).

As conical gears continue gaining more and more importance in the drivetrain industry, extensive investigations have been carried out — especially on geometry, contact analysis and bending stress. Mitome et al. (Refs. 6–7) conducted theoretical and experimental research concerning design and manufacturing methods of conical involute gears. Brauer (Ref. 8), Innocenti (Ref. 9) and Ohmachi et al. (Ref. 10) derive mathematical models based on conjugate flank meshing and contact characteristics, as well as on contact analysis using geometric shapes without modifications and practice-based deviations. Zhu et al. (Ref. 11) proposed a pitch cone design theory with regard to misalignments on tooth contact behavior of crossed beveloid gears. Fuentes et al. (Ref. 12) optimized the tooth profile of beveloid gears with regard to improved bearing contact and reduced noise excitation.

The exemplarily described investigations and methods all have in common the fact that the calculated geometry is based on an idealized basic rack profile or cutting / grinding tool simulation without modifications or deviations. The three-dimensional geometry of the beveloid flanks can be computed by means of differential geometry methods, as they are documented by Litvin et al. (Refs. 13–14). A practice-oriented geometry by a cutting or grinding process will lead to deviations on the tooth flanks that in fact influence the meshing characteristics (Refs. 15–16).

An elementary, fast and analytical method to determine the three-dimensional (non-modified) tooth geometry of a conical gear is to tilt a basic rack profile (Ref. 4) via cone angle θ (=



a) general basic rack profile b) 3D-meshing of the tilted basiic rack with a beveloid gear

Figure 4 On the geometry calculation of non-modified beveloid gears with a tilted rack.

addendum modification angle) and appliance of Equations 9–11. Therefore the geometry of the basic rack can be parametrized in sections y_{P1} and y_{Pn} with straight, polynomial and elliptic curves and the corresponding, outer unit normal vectors. To obtain the main geometric data of the conical gear, documented formula in Roth (Ref. 17) or Tsai (Ref. 18) can be used. Figure 4(b) clearly shows the calculated three-dimensional tooth geometry of a non-modified beveloid gear by a tilted basic rack profile (Fig. 4(a)). The calculated geometry does not take into account specific manufacturing deviations or modifications, as they occur in industrial practice.

Now, as we can feasibly calculate a reference geometry of a beveloid gear by a basic rack, we engage with the real manufacturing process of conical gearing.

The tool used for grinding (or cutting) is a worm with involute transverse (*ZI*) or straight normal section profile (*ZN*). All formulas to describe the main geometry of the tool are listed in DIN 3960 (Ref. 19) and DIN 8000 (Ref. 20). The geometry of the tool can be defined as an axial profile y_0 in its *y-z*-plane, exemplary for an involute worm flank as:

$$\tilde{y}_0(\lambda)=\begin{bmatrix}0\\ y_0\\ z_0\end{bmatrix}=\begin{bmatrix}0\\ \pm\frac{m_{x0}\cdot z_0}{2}\cdot(\tan\lambda-\lambda)\\ \frac{d_{b0}}{2\cdot\cos\lambda}\end{bmatrix},\lambda=\sqrt{\left(\frac{d}{d_{b0}}\right)^2-1}\text{ with }\frac{d_{Ff0}}{2}\le d\le\frac{d_{Fa0}}{2} \tag{16}$$

with axial module mx_0, number of threads z_0, base circle diameter db_0 and utilized root and tip diameters d_{Ff0} and d_{Fa0}. Further axial profiles for *ZA*, *ZN*, *ZK* or *ZC* worms can be calculated with the adapted methods of Predki (Ref. 21) and Octrue (Ref. 22), see (Ref. 3). To realize a root fillet contour, the tool tip edge is defined as a circle or elliptic arc (Ref. 3).

By definition of the axial profile, the three-dimensional geometry of the worm could be computed by screwing the axial profile along its axis to get a full parametrization. However, the usage of the basic law of gear kinematics with Equations 2–8 would lead to line contact between worm and beveloid gear. In fact, the meshing conditions between worm and workpiece are characterized by point of contact, comparable to crossed helical gears. To realize a fast and analytical simulation anyhow, a representation of the tool in its axial and parallel sections has proven to be most effective. Thus we are able to use Equations 9–12 to generate the beveloid tooth geometry by the axial (y_0) and parallel ($\tilde{y}_0$) sections of the worm tool — which is moving translational while tool and workpiece are rotating around their axes with a linked, angular velocity relation (Fig. 6c). The general parametrization of the parallel sections $\tilde{y}_0$ can be written as:

$$\tilde{y}_0(x_0)=\begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \tilde{z}_0\end{bmatrix}=\begin{bmatrix}x_0\\ y_0+\frac{m_{x0}\cdot z_0}{2}\cdot\Theta\\ z_0\cdot\cos\Theta\end{bmatrix}\text{ with }\Theta=\arcsin\left(\frac{x_0}{z_0}\right) \tag{17}$$

To realize the modification angle θ on industrial gear grinding or cutting machines, the tool is moved addendum by super-positioning the axial and radial feed, in addition to angular velocities of tool and workpiece. In the simulation model the relative position of tool and workpiece is arranged identically to an actual cutting/grinding machine. Taking into account the translational center distances a_0 and the rotation around swivel angle η, addendum modification angle θ and displacement angle ψ (which is needed due to kinematic restrictions of a cylindrical gear cutting/grinding machine [Refs. 15–16]), the tool-workpiece-position for the process kinematic is represented by:

$$Y_0=a_0+D_z(\psi)(D_y(\theta)(D_x(-\eta)(\tilde{y}_0))) \tag{18}$$

in which $\tilde{y}_0$ is the tooth shaping tool section (axial or parallel profile) in tool-based *0*-coordinate system, respectively, the

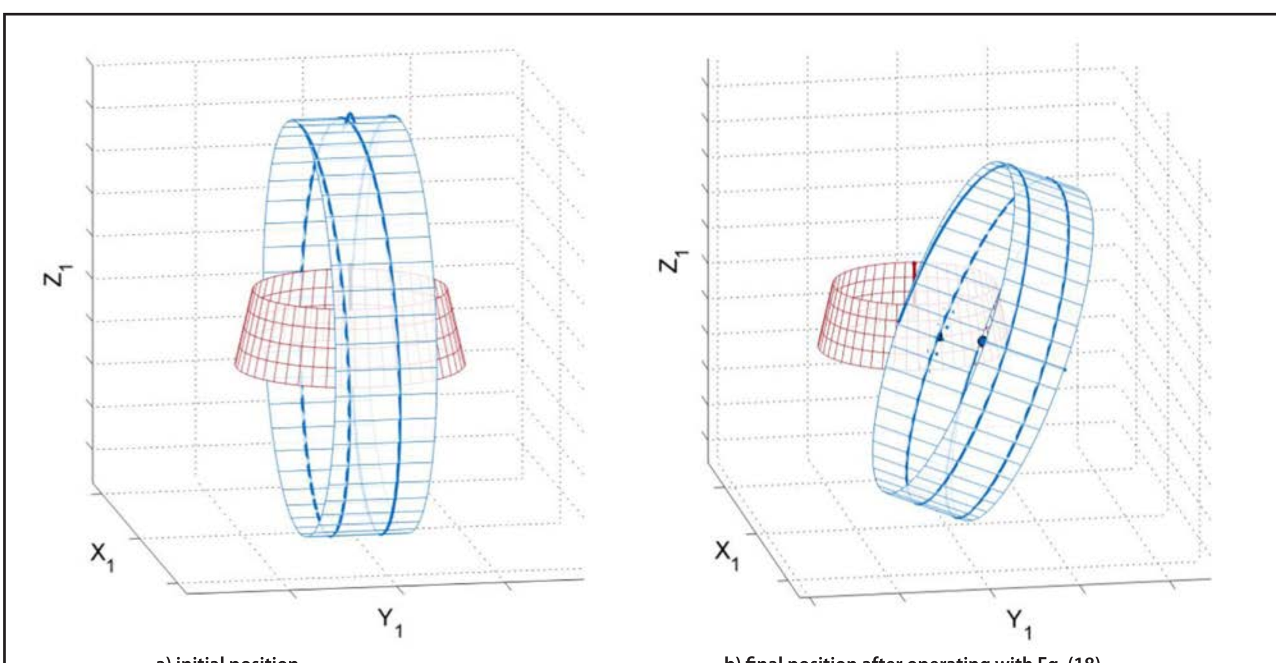


Figure 5 Calculating the relative position of cutting/grinding tool and workpiece.

tooth shaping tool-section Y_0 in workpiece-based global 1-coordinate system (Fig. 5).

With regard to the kinematic configuration of the beveloid grinding process, the operating point between tool and workpiece is shifted due to displacement angle ψ. Thus we get the adapted manufacturing parameters tool pitch angle γ', addendum modification angle θ and helix angle β' on the grinding machine (Ref. 16):

$$\sin\gamma' = \sin\gamma_{m0}\cdot\sin\theta \quad \tan\theta' = \frac{\tan\theta}{\cos\psi} \quad \tan\beta' = \frac{(\cos\gamma_{m0}+\sin\gamma_{m0}\cdot\tan\beta_0)\cdot\cos\psi-\cos\gamma'}{\sin\gamma_{m0}\cdot\cos\theta} \tag{19}$$

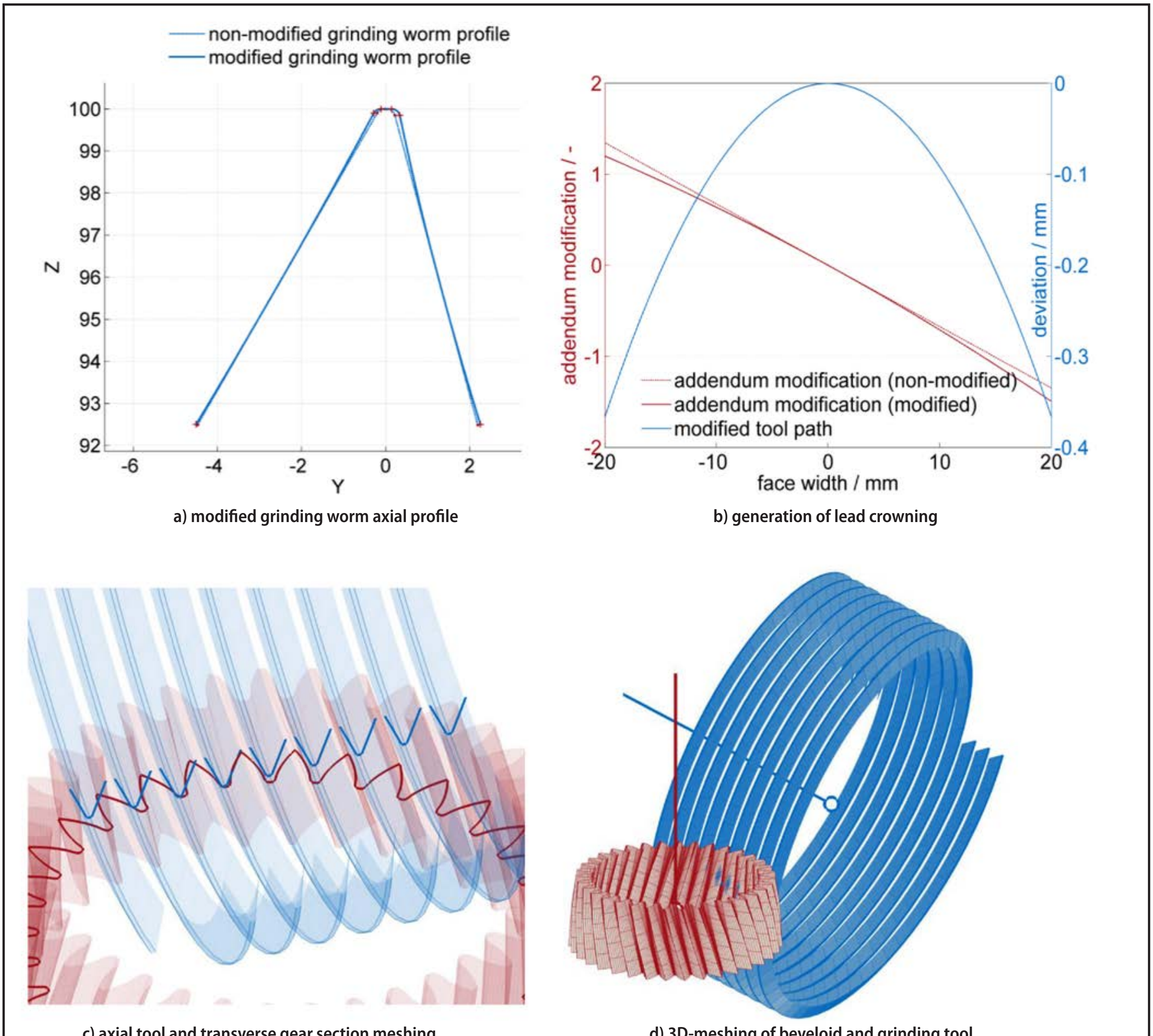
As a result, we cannot assure that the tooth shaping section is always determined by the tool axial profile y_0, but also have to take into account directly adjacent parallel sections $\tilde{y}_0$.

The synchronized motions of axial and radial tool feed of a real cutting/grinding machine can be converted into a spatial moving path $f(y_0, \tau)$ of the shaping tool section:

$$f(y_0,\tau) = a_1 + D_{b_1}\left(\frac{2\cdot\tan\beta_0}{d}\cdot\tau\right)(Y_0) + \tau\cdot b_1 \text{ with } 0\le\tau\le b \tag{20}$$

with tool basic rack helix angle β_0, reference diameter d and face width b. Thus, for each face width segment, it is possible to calculate the two-dimensional tooth geometry of the beveloid gear, and eventually the complete three-dimensional tooth.

Furthermore, it is possible to design flank modification not only by modification of the axial tool section profile (Fig. 6a), but also by modification of the spatial tool travel path (Fig. 6b). For example, lead crowning of beveloid gears with the amount C_β can be generated with due consideration of so-called flank twist, which results owing to spatially extended path of contact (Ref. 15). Figure 6d illustrates the described context by an example of



a) modified grinding worm axial profile

b) generation of lead crowning

c) axial tool and transverse gear section meshing

d) 3D-meshing of beveloid and grinding tool

Figure 6 On the generation of modified beveloid tooth geometry by grinding.

a beveloid gear with geometry referred to in Table 1. The grinding tool used for generation has an asymmetric *ZN* profile.

By setting the addendum modification angle $\theta = 0°$, the presented calculation model is readily applicable to common cylindrical gears. To illustrate the described effects on flank twist, (Fig. 7) clearly shows the calculated difference between a non-modified, theoretical flank (generated by a virtual rack) and a diverging flank (with lead crowning, generated by an involute grinding worm) of a cylindrical helical gear, based on the main geometry in Table 1 with = 0°. Due to the local variation of the center distance between grinding worm and workpiece to generate lead crowning and, thus, a spatial contact line, the generating worm section removes varying amounts of material along the face width and profile direction of the workpiece, respectively. As a result, we get different amounts of profile angle variation on $z_1 = 0$ and $z_1 = b_2$ (z_1 = workpiece axis), which can be measured as flank twist.

Table 1 Exemplary geometry of beveloid gear and grinding tool

	Symbol	Value	Unit
number of threads (grinding worm)	z_0	3	-
number of teeth (work piece)	z_2	35	-
normal pressure angle	$\alpha_{n0L/R}$	30/15	°
normal module	m_{n0}	2.5	mm
helix angle (tool basic rack)	β_0	-18.70	°
addendum modification angle	θ	4.8	°
profile shift (at working plane)	x_{2C}	0.2387	-
face width	b_2	35	mm
working plane position	b_C	17.5	mm
lead crowning (ref. to working plane)	C_β	15	µm
tip Ø, grinding worm	d_{a0}	200	mm
reference Ø, grinding worm	d_{m0}	191.452	mm
root Ø, grinding worm	d_{f0}	184.972	mm
lead angle, grinding worm	γ_0	-2.2451	°

Practical Application: Rack-and-Pinion Meshing with Variable Ratio

Steering applications in modern automobiles often require a variable, non-uniform motion transfer between rotational steering wheel and translational steering rack. Changing the lane with high velocity requires a stable maneuvering behavior. On the other hand, with larger steering angles, steering characteristics should become more direct, for example, during a parking situation. To meet these requirements, a variable steering ratio in rack-and-pinion steering systems can be realized by varying the three-dimensional tooth geometry of the rack.

Therefore the theoretical tooth surface of the rack has to be calculated setting the basis for possible manufacturing. Ohmachi et al. (Ref. 23) presented a method to obtain the theoretical profile of the rack with involute pinion based on a numerical solution of Litvin's (Ref. 13) differential geometry formula of the basic law of gearing. Wou et al. (Ref. 24) researched the non-uniform gear rack meshing with regard to friction and efficiency. Vaujany et al. (Ref. 25), as well as Alexandru (Ref. 26), derived tooth profiles by kinematics simulation and varying transverse pressure angles.

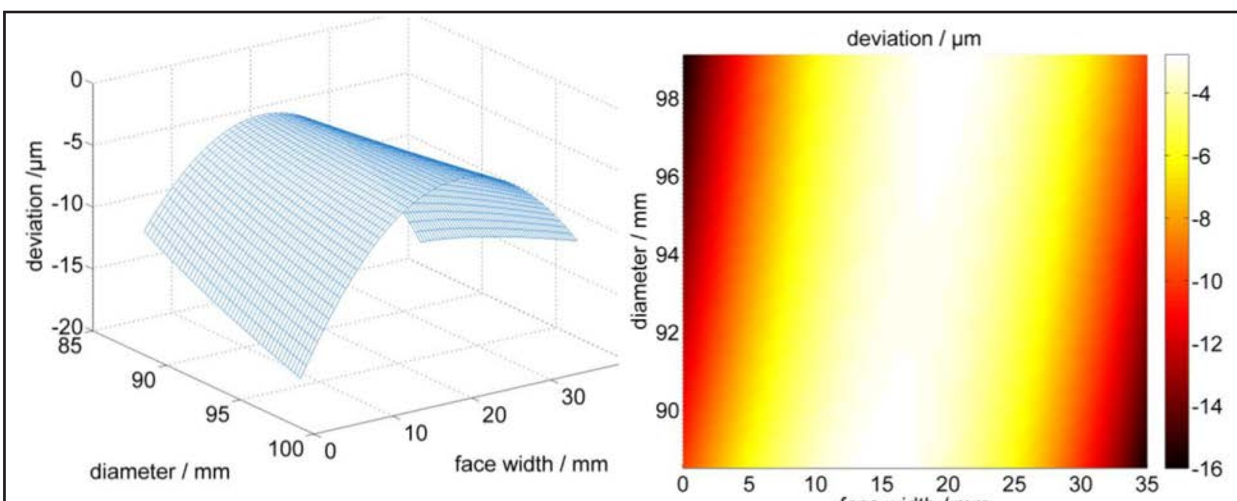
The described investigations need a set of information concerning pressure angles and working pitch radii or planes, etc. Without any previous knowledge about meshing characteristics, we start with a parametrization of the steering pinion, which is generally given by a helical involute surface $y(\sigma,\tau)$:

$$y(\sigma,\tau) = a + D_b\left(\frac{2\cdot\tan\beta}{d}\cdot\tau\right)\left(\frac{d_b}{2}\cdot\begin{bmatrix}\pm\sin\sigma \mp \sigma\cdot\cos\sigma\\ \cos\sigma + \sigma\cdot\sin\sigma\\ 0\end{bmatrix}\right) - \tau\cdot b \quad (21)$$

with the surface parameters:

$$\sigma = \sqrt{\left(\frac{d_y}{d_{Ff}}\right)^2 - 1} \text{ and } 0 \le \tau \le b \quad (22)$$

Alternatively, the pinion tooth shape can be calculated with the mentioned methods for generating cylindrical or conical involute gears.



a) flank twist due to lead crowning

b) deviation between theoretical flanks and flanks with lead crowning and twist effects

Figure 7 Generation of lead crowning and flank twist of a cylindrical gear.

Furthermore, the variable rack geometry depends on the transmission ratio, which can be given as rack gain to pinion rotation $\upsilon_1(\Delta_2)$, exemplarily expressed analytically by:

$$\upsilon_1(\Delta_2) = 60 - 10 \cdot e^{-0.00005\Delta_2^2} \tag{23}$$

see Figure 8 for $-360° \leq \Delta_2 \leq 360°$. The transmission ratio can also be defined by any other continuous function or derived practical specifications of the steering gearbox.

With knowledge of the pinion tooth shape and transmission function $\upsilon_1(\Delta_2)$, the desired three-dimensional rack geometry can be easily computed by applying Equations 13–15 for each local point of contact.

Figure 9a shows the geometry of rack-and-pinion meshing, illustrated in the pinion transverse plane. The axis-crossing angle in this example is $\Sigma = 82.3°$. The variation of the rack geometry profile in gain direction corresponds to the given steering ratio in Figure 8. What's more, due to the helical pinion shape the single transverse sections of the pinion are meshing time-delayed with the relevant rack shapes. Thus the geometry of the rack also changes in direction of the pinion face width (Fig. 9b). All relevant, main geometric parameters of pinion and rack are summarized in Table 2.

The identification of limits of the basic law of gearing for ex-

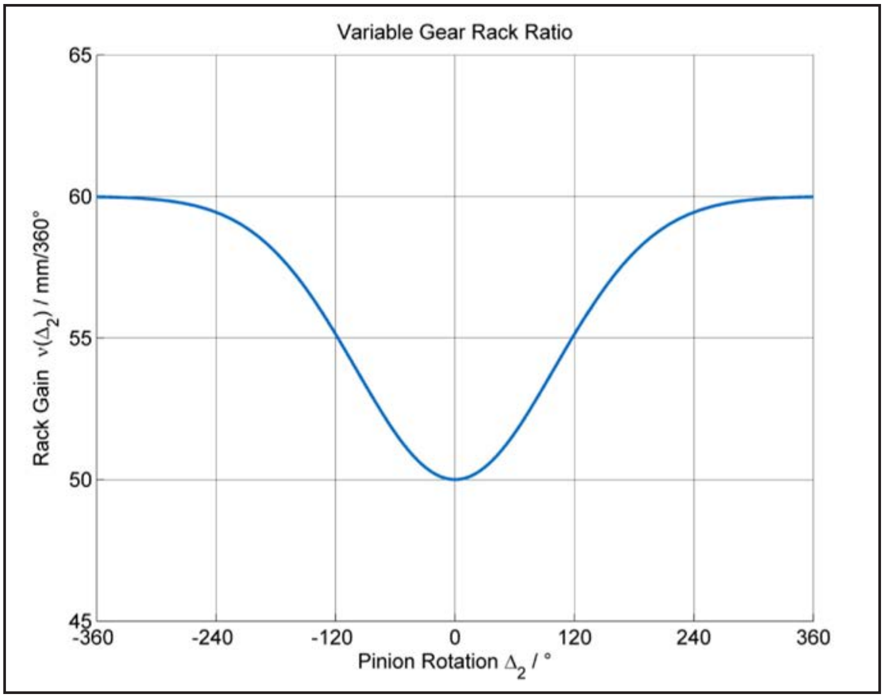


Figure 8 Variable gear rack ratio — as input by Equation 23.

Table 2 Exemplary geometry of pinion and rack with variable ratio

	Symbol	Value	Unit
number of teeth (rack)	z_0	16	-
number of teeth (pinion)	z_2	8	-
normal pressure angle	α_n	24	-
normal module	m_n	1.9	mm
helix angle	β_2	-25.0	°
profile shift	x_2	0.7509	-
pinion face width	b_2	36.5	mm
tip Ø, pinion	d_{a2}	11.0	mm
root Ø, pinion	d_{f2}	7.62	mm
axis crossing angle	Σ	82.3	°

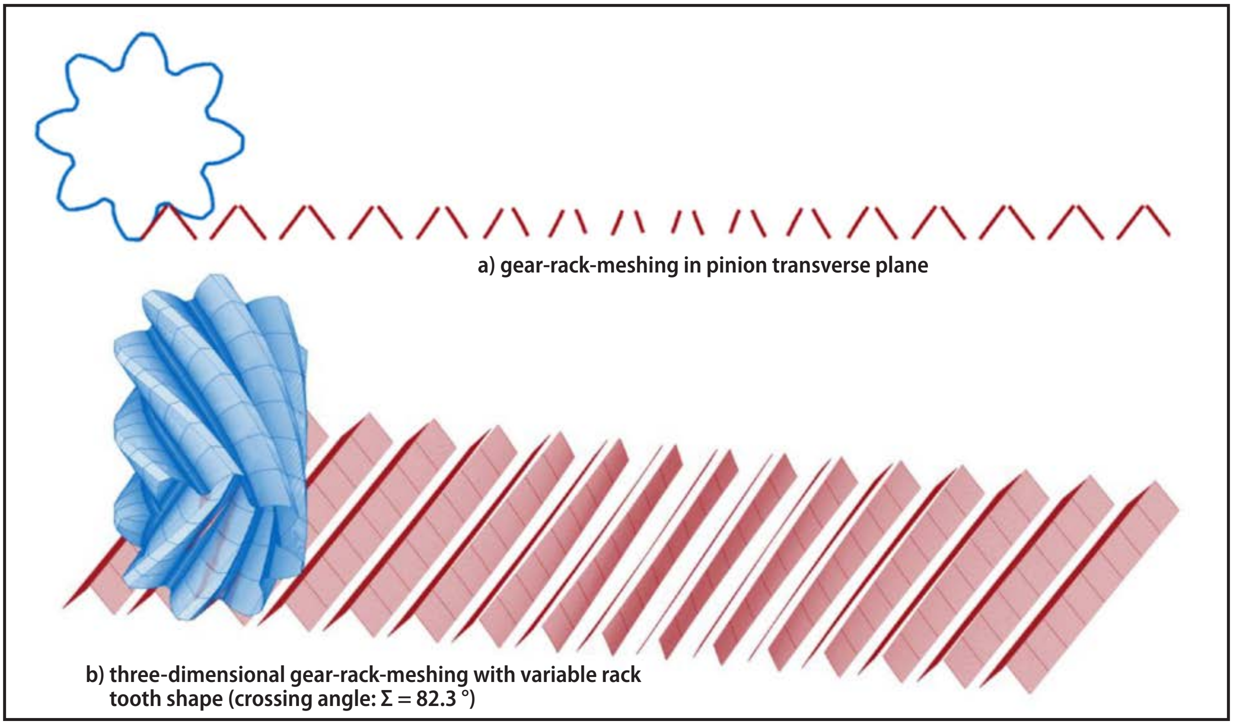


Figure 9 On the generation of the tooth geometry of a rack with variable gear ratio.

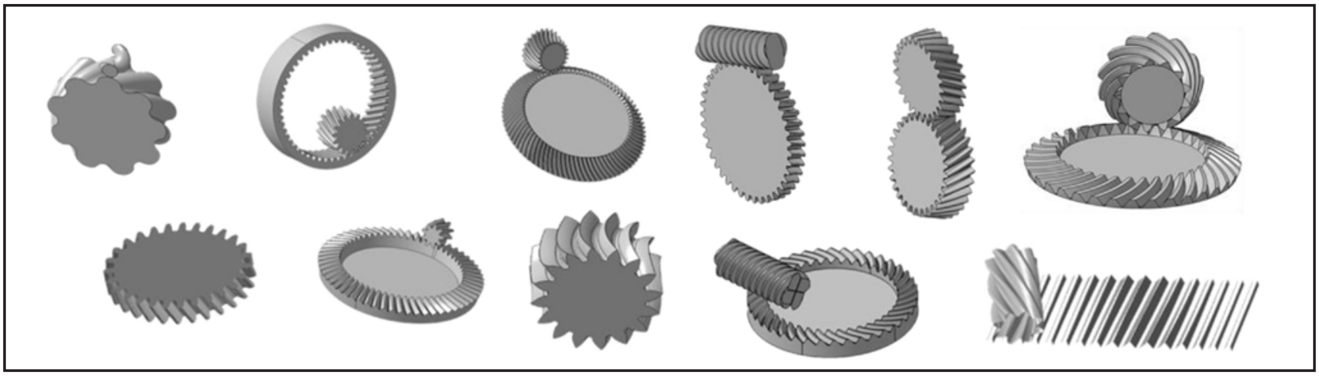


Figure 10 Exemplarily generated gear types — computed via integrated STEP-modeling.

treme transmission ratio, as well as limitation of the computed rack geometry (e.g., tip pointing, intersection with pinion root area, etc.), is subject to further geometric examinations.

Geometry Calculation Software "Flank Generator"

The described mathematical algorithms are implemented in software modules for the respective gear type then are summarized in the calculation software 'flank generator' for industrial usage to support design and optimization process of arbitrary gear types, and especially for non-standard gears and profiles. The following gear types can be computed (selective); (Fig. 10):

- (A)symmetric external and internal cylindrical involute gears
- Bevel and hypoid gears of selected manufacturing methods
- Worm gear drives
- Conical gears (face/crown gears and beveloid gears)
- Spiroid (worm face) gearings
- A multitude of spatial non-involute cylindrical gears (cycloid, W/N-, hybrid gears, EC-gearing, etc.
- Non-circular gears and rack-and-pinion meshing with variable ratio

For any further processing, an automated export of all computable three-dimensional geometries on the basis of a system-independent STEP-interface according to ISO 10303 (Ref. 27) by B-spline computation (Ref. 28) was developed (Fig. 10).

Conclusion

This paper exemplified a mathematical framework to calculate the tooth geometry of arbitrary gear and rack types. Thus, a fast and analytical tool is provided to compute and analyze the geometry of various gear applications, especially non-standard profiles that often cannot be examined by commercial industrial software; this is shown exemplarily in two practice-oriented examples by beveloid gears and variable ratio gear rack meshing.

With regard to state-of-the-art methods of general and particular description of gear shapes, Litvin et al. (Refs. 13–14), Mitome et al. (Refs. 6–7), and Ohmachi et al. (Refs. 23 and 10) derive mathematical models based on differential geometry methods, partially using geometric shapes without modifications and practice-based manufacturing deviations. These mathematical methods all have in common that the solution of the spatial basic law of gearing is complex to some extent, as the envelope to a family of contact lines has to be calculated by numerical solvers. In contrast, the presented conjugate algorithms for uniform motion transfer in this paper are characterized by full analytical, robust, and thus fast computable solutions.

Furthermore, the presented algorithms form the basis for further investigations concerning development and determination of the manufacturability of novel optimized tooth profiles, depending on the given applications. The described algorithms are summarized in implemented software modules for the particular gear types to support gear design process.

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